Solving the quadratic assignment problem by means of general purpose mixed integer linear programming solvers

Huizhen Zhang* Cesar Beltran-Royo† Liang Ma‡

19/04/2010

Abstract

The Quadratic Assignment Problem (QAP) can be solved by linearization, where one formulates the QAP as a mixed integer linear programming (MILP) problem. On the one hand, most of these linearization are tight, but hardly exploited within a reasonable computing time because of their size. On the other hand, Kaufman and Broeckx formulation [1] is the smallest of these linearizations, but very weak. In this paper, we analyze how Kaufman and Broeckx formulation can be tightened to obtain better QAP-MILP formulations. As we show in our numerical experiments, these tightened formulations remain small but computationally effective in order to solve the QAP by means of general purpose MILP solvers.

Key words: quadratic assignment problem, mixed integer linear programming.

1. Introduction

In 1957, the Quadratic Assignment Problem (QAP) was introduced by Koopmans and Beckmann [2] as a mathematical model for the location of a set of indivisible economic activities. In general, QAP can be described as a one-to-one assignment problem of n facilities to n locations, which minimizes the sum of the total quadratic interaction cost, the flow between the facilities multiplied with their distances, and the total linear cost associated with allocating a facility to a certain location. Consider the set $N = \{1, 2, \cdots, n\}$ and three $n \times n$ matrices $F = (f_{ij})$, $D = (d_{ij})$ and $C = (c_{ij})$, the quadratic assignment problem with coefficient matrices F, D and C, shortly denoted by QAP, can be stated as follows:

*zhzzywz@gmail.com, Statistics and Operations Research, Rey Juan Carlos University, Madrid, Spain.

†Corresponding author: cesar.beltran@urjc.es, Statistics and Operations Research, Rey Juan Carlos University, Madrid, Spain.

‡maliang@usst.edu.cn, School of Management,University of Shanghai for Science and Technology,Mail box 459, 516 Jungong Road, Shanghai 200093,China

$$\min_{x\in X} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ik}d_{jl}\ x_{ij}x_{kl} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{1.1}$$

where

$$X = \Big\{ x \ \mid \quad \sum_{j=1}^{n} x_{ij} = 1 \qquad i \in N \tag{1.2}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j \in N \tag{1.3}$$

$$x_{ij} \in \{0,1\} \qquad i,j \in N \Big\}, \tag{1.4}$$

f_{ik} denotes the amount of flow between facilities i and k, d_{jl} denotes the distance between locations j and l, and c_{ij} denotes the cost of locating facility i at location j. $x_{ij} = 1$ if facility i is assigned to location j, otherwise, $x_{ij} = 0$.

In [3] a more general expression of (1.1) was introduced by using a four-dimensional array $\hat{q}_{ijkl}$ instead of the flow-distance products $f_{ik}d_{jl}$:

$$\min_{x\in X} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \hat{q}_{ijkl}\ x_{ij}x_{kl} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}. \tag{1.5}$$

Obviously, the linear terms can be eliminated by setting $q_{ijij} := \hat{q}_{ijij} + c_{ij}$:

$$\min_{x\in X} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} q_{ijkl}\ x_{ij}x_{kl}. \tag{1.6}$$

Without loss of generality, we can assume that q_{ijkl} are nonnegative. If they are negative, we add a sufficiently large constant to all q_{ijkl}, which does not change the optimal permutation and increase the objective function by n^2 times the added constant.

Over the years, the QAP has drawn the researcher's attention worldwide and extensive research has been done. From the theoretical point of view, it is because of the high computational complexity: QAP is NP-hard, and even finding an ε-approximate solution is a hard problem [4]. Moreover, many well-known classical combinatorial optimization problems such as the traveling salesman problem, the graph partitioning problem, the maximum clique problem can be reformulated as special cases of the QAP, see [5] and [6] for details. From the practical point of view, it is because of the diversified applications of the QAP. The QAP has been applied in many fields such as backboard wiring [7], typewriter keyboards and control panels design [8], scheduling [9], numerical analysis [10], storage-and-retrieval [11], and many others. More advances in theoretical aspects, solution methods and applications of the QAP can be found in [5, 12, 13, 14, 15, 16, 17, 18].

One common approach to solve the QAP is to 'linearize' it, that is, reformulate it as a pure or mixed integer linear programming problem. Lawler [3] replaced the quadratic terms $x_{ij}x_{kl}$ in the objective function by n^4 variables $y_{ijkl} = x_{ij}x_{kl}$. **More details of this reformulation linearization for the QAP can also be found in [19, 20, 21]**. The main drawback of this approach is the huge number of variables. Kaufman and Broeckx [1] proposed a mixed integer linearization with n^2 binary variables and n^2 real variables (see Section 2.2.). Although this is the smallest QAP linearization, its LP relaxation is known to be usually weak.

Recently, Xia and Yuan [22, 23, 24] tightened Kaufman-Broeckx formulation, basically, by introducing new constraints based on the Gilmore-Lawler constants (see Section 2.3.). In this paper we will concentrate on linerizations derived from the Kaufman-Broeckx formulation, which we will call the Kaufman-Broeckx (KB) family of formulations.

The objective of this paper is to study the performance of the KB family of formulations when one uses a general purpose mixed integer linear programming solver to solve the QAP. This objective will be developed as follows. In section 2, we will review the Gilmore-Lawler bound, the Kaufman-Broeckx linearization and the Xia-Yuan linearization. In section 3, we will enhance the KB family of formulations with a new QAP linearization. In section 4, on the one hand we will show that the Kaufman-Broeckx formulation is the weakest possible QAP linearization and on the other hand, we will study the LP relaxation of the new linearization. In section 5 we will analyze the computational performance of the KB family of formulations.

2. The Kaufman-Broeckx family of linearizations

Here we shortly review the Gilmore-Lawler bound, and the two current smallest QAP linearizations, namely, Kaufman-Broeckx linearization and Xia-Yuan linearization. As we already said, Xia and Yuan have derived their formulation from Kaufman-Broeckx formulation. In this paper we will study this type of formulations which we will call the Kaufman-Broeckx (KB) family of formulations.

2.1. Gilmore-Lawler bound

The Gilmore-Lawler bound was derived by Gilmore [25] and Lawler [3]. Consider the Lawler QAP (1.5) with coefficient matrix $Q = (q_{ijkl})$. For each pair $i, j \in N$, solve the following linear assignment problem (LAP) and denote its optimal value as l_{ij}:

$$\min \sum_{\substack{k=1 \\ k\neq i}}^{n} \sum_{\substack{l=1 \\ l\neq j}}^{n} q_{ijkl}\ x_{kl} \tag{2.7}$$

$$\sum_{k\neq i}^{n} x_{kl} = 1, \qquad l \in N \ \ and \ \ l \neq j \tag{2.8}$$

$$\sum_{l\neq j}^{n} x_{kl} = 1, \qquad k \in N \ \ and \ \ k \neq i \tag{2.9}$$

$$x_{kl} \in \{0, 1\}, \qquad k, l \in N \ \ and\ k \neq i,\ l \neq j \tag{2.10}$$

The Gilmore-Lawler bound for the QAP is given by the optimal value of the LAP of size n with cost matrix $(l_{ij} + q_{ijij})$:

$$GLB: \qquad \min_{x\in X} \sum_{i=1}^{n} \sum_{j=1}^{n} (l_{ij} + q_{ijij}) x_{ij} \tag{2.11}$$

2.2. Kaufman-Broeckx linearization

Kaufman and Broeckx [1, 26] introduced n^2 continuous variables

$$\tilde{z}_{ij} = x_{ij} \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl} x_{kl}$$

and $n^2 + 2n$ constraints, to derive the following QAP linearization:

$$KBL: \quad \min_{x \in X} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{z}_{ij} \tag{2.12}$$

$$\text{s. t.} \quad \tilde{z}_{ij} \geq \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl} x_{kl} - \tilde{a}_{ij}(1 - x_{ij}) \qquad i, j \in N \tag{2.13}$$

$$\tilde{z}_{ij} \geq 0 \qquad i, j \in N \tag{2.14}$$

where

$$\tilde{a}_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl}. \tag{2.15}$$

2.3. Xia-Yuan linearization

Xia and Yuan [22, 23, 24] tightened Kaufman and Broeckx formulation, basically, by introducing new constraints based on the Gilmore-Lawler constants l_{ij}:

$$XYL: \quad \min_{x \in X} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} (\bar{z}_{ij} + q_{ijij} x_{ij}) \tag{2.16}$$

$$\text{s. t.} \quad \bar{z}_{ij} \geq \sum_{\substack{k=1 \\ k \neq i}}^{n} \sum_{\substack{l=1 \\ l \neq j}}^{n} q_{ijkl} x_{kl} - \bar{a}_{ij}(1 - x_{ij}) \qquad i, j \in N \tag{2.17}$$

$$\bar{z}_{ij} \geq l_{ij} x_{ij} \qquad i, j \in N \tag{2.18}$$

where

$$\bar{a}_{ij} = \max\{\sum_{\substack{k=1 \\ k \neq i}}^{n} \sum_{\substack{l=1 \\ l \neq j}}^{n} q_{ijkl} x_{kl} : \ x_{kl} \text{ satisfy } (2.8) - (2.10)\} \text{ for } i, j \in N \tag{2.19}$$

In [23] it is also proved that Xia-Yuan bound is stronger than the Gilmore-Lawler bound:

Theorem 2.1 *Let f^*_{RXYL} be the optimal solution of the LP relaxation of (XYL) and f^*_{GLB} the optimal solution of (GLB). Then:*

$$f^*_{RXYL} \geq f^*_{GLB}.$$

3. A new linearization in Kaufman-Broeckx family

Now we present another way to tighten Kaufman-Broeckx formulation by using the Gilmore-Lawler constants l_{ij}. This formulation turns out to be similar to Xia-Yuan formulation, as we will see in Proposition 3.1. However, this new formulation may obtain slightly tighter LP bounds (see in Table 4, instances labeled by 'Bur'). This new QAP formulation, which we call Gilmore-Lawler Linearization (GLL), is as follows:

$$GLL: \quad \min_{x\in X} \sum_{i=1}^{n}\sum_{j=1}^{n}[z_{ij} + (q_{ijij} + l_{ij})x_{ij}] \tag{3.20}$$

$$\text{s. t.} \quad z_{ij} \geq \sum_{k=1}^{n}\sum_{l=1}^{n} q_{ijkl}x_{kl} - a_{ij}(1 - x_{ij}) - (l_{ij} + q_{ijij})x_{ij} \qquad i,j \in N \tag{3.21}$$

$$z_{ij} \geq 0 \qquad i,j \in N \tag{3.22}$$

where

$$a_{ij} = \max\left\{\sum_{k=1}^{n}\sum_{l=1}^{n} q_{ijkl}x_{kl} : \quad x \in X\right\} \tag{3.23}$$

Theorem 3.2 *To solve GLL is equivalent to solve the QAP.*

Proof: Let us consider $F_{QAP}, f_{QAP}(x)$ and f^*_{QAP}, the QAP feasible set, the QAP objective function and the QAP optimum, respectively. Analogously, we consider $F_{GLL}, f_{GLL}(x,z)$ and f^*_{GLL}.

Firstly, let us see that any given solution $(x,z) \in F_{GLL}$ can be projected to a solution $x \in F_{QAP}$ with the same objective value. We consider the following two cases:

i). If $x_{ij} = 1$, then $z_{ij} \geq \sum_{k,l=1}^{n} q_{ijkl}x_{kl} - (l_{ij}+q_{ijij})x_{ij}$ by (3.21). Considering the definition of f_{GLL}, we have

$$z_{ij} + (l_{ij} + q_{ijij})x_{ij} = \sum_{k,l=1}^{n} q_{ijkl}x_{kl} = \sum_{k,l=1}^{n} q_{ijkl}x_{kl}x_{ij}.$$

ii). If $x_{ij} = 0$, then $z_{ij} + (l_{ij} + q_{ijij})x_{ij} = 0$ by (3.21), (3.22), and the definition of f_{GLL}. Therefore, we have

$$z_{ij} + (l_{ij} + q_{ijij})x_{ij} = \sum_{k,l=1}^{n} q_{ijkl}x_{kl}x_{ij}.$$

Thus, we have seen that $f_{GLL}(x,z) = f_{QAP}(x)$ as we wanted to see. This implies, $f^*_{GLL} \geq f^*_{QAP}$.

Secondly, let us see that any given solution $x \in F_{QAP}$ can be lifted to a solution $(x,z) \in F_{GLL}$ with the same objective value. Given a $x \in F_{QAP}$, we define $z_{ij} = \sum_{k,l=1}^{n} q_{ijkl}x_{kl} - (l_{ij} + q_{ijij})$ if $x_{ij} = 1$ and $z_{ij} = 0$, otherwise. Obviously, $(x,z) \in F_{GLL}$. Furthermore, for any $i,j \in N$,

$$z_{ij} + (l_{ij} + q_{ijij})x_{ij} = x_{ij}\sum_{k,l=1}^{n} q_{ijkl}x_{kl} = \sum_{k,l=1}^{n} q_{ijkl}x_{kl}x_{ij},$$

whether $x_{ij} = 1$ or $x_{ij} = 0$. Thus, we have seen that $f_{QAP}(x) = f_{GLL}(x, z)$ as we wanted to see. This implies, $f^*_{QAP} \geq f^*_{GLL}$ and therefore $f^*_{QAP} = f^*_{GLL}$

All in all, we have proved that if (x^*, z^*) is optimal for GLL, then x^* is optimal for QAP.

■

Proposition 3.1 *GLL can be written as*

$$\min_{x \in X} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \widehat{z}_{ij} \tag{3.24}$$

$$s.\ t. \quad \widehat{z}_{ij} \geq \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl} x_{kl} - a_{ij}(1 - x_{ij}) \qquad i, j \in N \tag{3.25}$$

$$\widehat{z}_{ij} \geq (q_{ijij} + l_{ij}) x_{ij} \qquad i, j \in N \tag{3.26}$$

Proof: Inequalities (3.21) and (3.22) are equivalent to the following inequalities, respectively

$$z_{ij} + (q_{ijij} + l_{ij}) x_{ij} \geq \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl} x_{kl} - a_{ij}(1 - x_{ij}) \qquad i, j \in N \tag{3.27}$$

$$z_{ij} + (q_{ijij} + l_{ij}) x_{ij} \geq (q_{ijij} + l_{ij}) x_{ij} \qquad i, j \in N \tag{3.28}$$

To finish the proof we only need to define $\widehat{z}_{ij} = z_{ij} + (q_{ijij} + l_{ij}) x_{ij}$, and replace $z_{ij} + (q_{ijij} + l_{ij}) x_{ij}$ by $\widehat{z}_{ij}$ in (3.20), (3.27) and (3.28). ■

4. Bounds based on the Kaufman-Broeckx family of formulations

It is well known that the LP relaxation of the Kaufman-Broeckx linearization gives a poor QAP bound. In this section, we prove that it gives the worst possible QAP bound, that is, 0. We also prove that much better bounds can be obtained by combining Kaufman-Broeckx linearization with Gilmore-Lowler constants l_{ij}, as in linerizations XYL and GLL. We name $RGLL$, $RXYL$ and $RKBL$ the LP relaxations of GLL, XYL and KBL, respectively.

Theorem 4.3 *Let f^*_{RKBL} be the optimal value of the LP relaxation of the Kaufman-Broeckx linearization. Then*

$$f^*_{RKBL} = 0.$$

Proof: Let F_{RKBL} be the feasible set of $RKBL$. Considering that the objective value of $RKBL$ is always positive, to prove this result, it is enough to see that there exists one solution $(x^0, \tilde{z}^0) \in F_{RKBL}$ such that $f_{RKBL}(x^0, \tilde{z}^0) = 0$.

As such point, we take $x^0_{ij} = \frac{1}{n}$ and $\tilde{z}^0_{ij} = 0$ for all $i, j \in N$. Firstly, let us see that $(x^0, \tilde{z}^0) \in F_{RKBL}$. Obviously, $x^0 \in X$ and $\tilde{z}^0_{ij} \geq 0$. Furthermore, for each $i, j \in N$ and for any

$n > 2$, we have

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{l=1}^{n} q_{ijkl}\frac{1}{n} - \tilde{a}_{ij}(1-\frac{1}{n}) \\
&= \frac{1}{n}(\sum_{k=1}^{n}\sum_{l=1}^{n} q_{ijkl} - \tilde{a}_{ij}(n-1)) \\
&= \frac{1}{n}(\tilde{a}_{ij} - \tilde{a}_{ij}(n-1)) \\
&= \frac{2-n}{n}\;\tilde{a}_{ij} \\
&< 0.
\end{aligned}$$

Therefore, $(x^0, \tilde{z}^0)$ also satisfies (2.13) and thus $(x^0, \tilde{z}^0) \in F_{RKBL}$. On the other hand, it is clear that $f_{RKBL}(x^0, \tilde{z}^0) = 0$. ■

Theorem 4.4 *Let f^*_{RGLL}, f^*_{GLB} and f^*_{RKBL} be the optimal objective values of RGLL, GLB and RKBL, respectively. Then*

$$f^*_{RGLL} \geq f^*_{GLB} \geq f^*_{RKBL} \tag{4.29}$$

Proof: We denote by *RGLB* the LP relaxation of *GLB*. Considering that *GLB* is a linear assignment problem we have that $f^*_{RGLB} = f^*_{GLB}$. One can write *RGLB* as

$$\begin{aligned}
RGLB: \quad & \min \quad \sum_{i=1}^{n}\sum_{j=1}^{n} z'_{ij} & & (4.30)\\
& \text{s. t.} \quad (1.2) - (1.3) & & \\
& \quad z'_{ij} \geq (l_{ij} + q_{ijij})x_{ij} & i,j \in N \quad & (4.31)\\
& \quad 0 \leq x_{ij} \leq 0 & i,j \in N \quad & (4.32)
\end{aligned}$$

Thus, to prove this theorem, we only need to prove $f^*_{RGLL} \geq f^*_{RGLB}$ and $f^*_{RGLB} \geq f^*_{RKBL}$.

i). Firstly, let us see $f^*_{RGLL} \geq f^*_{RGLB}$. By Proposition 3.1 it is clear that *RGLB* is a relaxation of *RGLL*. Therefore $f^*_{RGLL} \geq f^*_{RGLB}$.

ii). Secondly, let us see $f^*_{RGLB} \geq f^*_{RKBL}$. By constraints (4.31) we have $z'_{ij} \geq 0$ for $i, j \in N$. Therefore, $f^*_{RGLB} \geq 0$ and by Theorem 4.3 we can conclude that $f^*_{RGLB} \geq f^*_{RKBL}$. ■

Lemma 4.1 *For a given QAP instance, if $f_{ii} = 0$ and $d_{ii} = 0$ for all $i \in N$, then*

$$c_{ij} + \sum_{\substack{k=1\\k\neq i}}^{n}\sum_{\substack{l=1\\l\neq j}}^{n} q_{ijkl} = \sum_{k=1}^{n}\sum_{l=1}^{n} q_{ijkl} \qquad i,j \in N \tag{4.33}$$

$$c_{ij} + \bar{a}_{ij} = a_{ij} \qquad i,j \in N \tag{4.34}$$

Proof: Trivial.

Proposition 4.2 *Given a QAP instance defined by flow matrix F and distance matrix D, if $f_{ii} = 0$ and $d_{ii} = 0$ for all $i \in N$, then XYL can be written as*

$$\min_{x \in X} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \widehat{z}_{ij} \tag{4.35}$$

$$\text{s. t.} \quad \widehat{z}_{ij} \geq \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl} x_{kl} - a_{ij}(1 - x_{ij}) + c_{ij}(1 - x_{ij}) \qquad i, j \in N \tag{4.36}$$

$$\widehat{z}_{ij} \geq (q_{ijij} + l_{ij}) x_{ij} \qquad i, j \in N \tag{4.37}$$

Proof: If $f_{ii} = 0$ and $d_{ii} = 0$ for all $i \in N$, we have $q_{ijij} = c_{ij}$ and $\sum_{\substack{k=1 \\ k \neq i}}^{n} q_{ijkj} + \sum_{\substack{l=1 \\ l \neq j}}^{n} q_{ijil} = 0$. Furthermore, constraint (2.17) can be rewritten as follows

$$(2.17) \iff \bar{z}_{ij} + c_{ij} x_{ij} \geq \sum_{\substack{k=1 \\ k \neq i}}^{n} \sum_{\substack{l=1 \\ l \neq j}}^{n} q_{ijkl} x_{kl} - \bar{a}_{ij}(1 - x_{ij}) + c_{ij} x_{ij}$$

$$\text{(By Lemma 4.1)} \tag{4.38}$$

$$\iff \bar{z}_{ij} + q_{ijij} x_{ij} \geq \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl} x_{kl} - (a_{ij} - c_{ij})(1 - x_{ij}) \tag{4.39}$$

$$\iff \bar{z}_{ij} + q_{ijij} x_{ij} \geq \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl} x_{kl} - a_{ij}(1 - x_{ij}) + c_{ij}(1 - x_{ij}) \tag{4.40}$$

Similarly, constraint (2.18) can be rewritten as follows

$$\begin{aligned} (2.18) &\iff \bar{z}_{ij} + c_{ij} x_{ij} \geq l_{ij} x_{ij} + c_{ij} x_{ij} \\ &\iff \bar{z}_{ij} + q_{ijij} x_{ij} \geq (l_{ij} + q_{ijij}) x_{ij} \end{aligned} \tag{4.41}$$

Now, we can replace $\bar{z}_{ij} + q_{ijij} x_{ij}$ by $\widehat{z}_{ij}$ in (2.16), (4.40) and (4.41), to obtain the following formulation equivalent to XYL:

$$\min_{x \in X} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \widehat{z}_{ij} \tag{4.42}$$

$$\text{s. t.} \quad \widehat{z}_{ij} \geq \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl} x_{kl} - a_{ij}(1 - x_{ij}) + c_{ij}(1 - x_{ij}) \qquad i, j \in N \tag{4.43}$$

$$\widehat{z}_{ij} \geq (l_{ij} + q_{ijij}) x_{ij} \qquad i, j \in N \tag{4.44}$$

■

Theorem 4.5 *Given a QAP instance defined by flow matrix F and distance matrix D, if $f_{ii} = 0$ and $d_{ii} = 0$ for all $i \in N$, then*

$$f^*_{RXYL} \geq f^*_{RGLL},$$

where f^*_{RXYL} and f^*_{RGLL} are the optimal objective values of $RXYL$ and $RGLL$, respectively.

Proof: From Propositions 3.1 and 4.2 it is clear that the feasible set of formulation GLL contains the feasible set of formulation XYL. Furthermore, the two formulations have the same objective function. ■

Corollary 4.1 *Given a QAP instance defined by flow matrix F and distance matrix D, if $f_{ii} = 0, d_{ii} = 0$ and $c_{ij} = 0$ for all $i, j \in N$, then*

$$f^*_{RXYL} = f^*_{RGLL}.$$

Proof: From Propositions 3.1 and 4.2.

5. Numerical experiments

The objective of this section is to assess the effectiveness of the small QAP linearizations presented in this paper. We present the experimental results obtained with formulations GLL, KBL and XYL used to solve a representative set of instances from QAPLIB [27]. Furthermore, we compare these results with formulation $IPQAPR-III$ presented in [28]. To the best of our knowledge, formulation $IPQAPR-III$ is the best one in order to solve the QAP by a general purpose MILP solver.

The CPU time limit was set to 14400 seconds. The experiments were conducted on a laptop with a processor Intel Core Duo 2.80GHz and with 3.95 GB of RAM. Cplex 11.2 (default parameters) interfaced with Matlab 2008b [29] was used to solve the QAP instances. We have to take into account that, as reported in [28], the $IPQAPR-III$ results were obtained by using a laptop with a processor Intel Pentium M 1.70GHz and with 1.23 GB of RAM. Cplex 9.0 (default parameters) interfaced with Matlab 7.0 was used to solve the QAP instances.

In table 1 we describe the instances used in our test, namely, the size of the instances, the number of variables x_{ij}, y_{ij} and z_{ij}, and the number of constraints. In these experiments, we studied two groups of instances: sparse instances (Chr18a-Scr20) and dense ones (Bur26a-Nug30). In table 1, we also report the *density of the flow matrix* $F = (f_{ij})$ (DFM), which is defined as the proportion of non-zero elements in the matrix (in %).

In table 2 we report the quality of the MILP solution computed by Cplex. We present its objective function (Cost), its brand and bound gap as computed by Cplex $B\&B\ Gap = \frac{Cost-B\&BBound}{10^{-10}+|Cost|} \times 100\%$ and its optimality gap $Opt.\ Gap = |\frac{Cost-Opt.\ Cost}{Opt.\ Cost}| \times 100\%$, where $Opt.\ Cost$ denotes the optimal or best known **cost** reported in literature.

For the 31 tested instances, there are 17, 13, 20 and 9 near optimal solutions ($Opt.\ Gap \leq 1\%$) computed by GLL, KBL, XYL and $IPQAPR-III$, respectively. By using the same four formulations, for 6, 1, 6 and 6 instances, respectively, optimality was proved ($B\&B.\ Gap = 0\%$). Therefore, these results also imply that GLL and XYL are the improvements of the KBL. We observe that, regarding the solution quality, formulation XYL outperforms the other three formulations, closely followed by formulation GLL.

In table 3 we report the CPU time spent by Cplex to solve the corresponding MILP and the number of branch and bound nodes. In the first group of instances (sparse instances) we observe that, regarding the CPU time, formulation $IPQAPR-III$ can compete with formulations GLL and XYL ($IPQAPR-III$ was designed to exploit the sparsity of the QAP cost matrix). We also observe that in formulation GLL, and especially in formulation KBL, Cplex stops because an 'out of memory'. Those results have been obtained by using Cplex default parameters (the results obtained by tuning Cplex memory management parameters have been similar). Therefore, regarding the CPU time and memory requirements, formulation XYL outperforms the other three formulations.

In table 4 we present the quality of the LP bound and the CPU time required to obtain it. Furthermore, we compute the $LPgap = \frac{Opt.\ Cost - LPcost}{Opt.\ Cost} \times 100\%$, where $Opt.\ Cost$ is the optimal or best known objective value. In this respect, formulations GLL and XYL clearly outperform the other two formulations. As proved in Theorem 4.3, KBL always gives the worst possible gap and formulation $IPQAPR-III$ shows much longer CPU times for dense instances. Finally, we observe that formulation GLL may obtain slightly tighter bounds than formulation XYL, as it is the case for the 'Bur' group of instances.

6. Concluding Remarks

The main objective of this paper is to analyze the possibility of solving the Quadratic Assignment Problem (QAP) by means of general purpose mixed integer linear programming solvers, as for example CPLEX. The main conclusion is that for this purpose the Kaufman-Broeckx family of formulations, especially Xia-Yuan formulation, is the most effective. We have considered three members in this family. The first member of this family is the Kaufman-Broeckx linearization (KBL). As it is known, this formulation is the smallest one but very weak in general. The second member is the Xia-Yuan linearization (XYL), which tightens KBL by introducing new constraints based on the Gilmore-Lawler constants l_{ij}. We have introduced the third member of the Kaufman-Broeckx family which we have called the Gilmore-Lawler linearization (GLL).

From a theoretical point of view first, we have proved that the KBL LP bound is always 0, the worse possible bound obtained by linearization. Second, we have proved that the bound given by GLL, the new QAP formulation, is stronger than the Gilmore-Lawler bound. Third, we have proved, that under particular conditions, XYL is stronger than GLL. However, in general, we cannot say that XYL is stronger than GLL: in Table 2 we have observed that for the 'Bur' instances the GLL LP bound is stronger than the XYL one.

From a numerical point of view, we have compared Kaufman-Broeckx family of formulations to formulation $IPQAPR - III$, a state of the art formulation used to solve the QAP with a general purpose MILP. in Table 1 we have observed that the Kaufman-Broeckx family of formulations has a very reduced size. In Table 2, we have observed that GLL and specially XYL, obtain very good feasible solutions (the optimality error is under 1% in 70% of the cases). Compared to other linearizations, the great advantage of GLL and XYL, is that they have a moderate LP gap (average around 20%) and that the LP solving time is very small (around 2 seconds) as shown in Table 4. The main drawback of these two formulations, specially in the case of GLL, is the fast saturation of the branch-and-bound tree (Table 3).

Acknowledgements

Thanks are due to the support of the National Natural Science Foundation of China (Grant No.70871081), and to the support from MAEC-AECID fellowship from the Spanish government. We wish to thank the Faculty of Sciences of the University of Lisbon for providing the software Cplex 11.2. We also thank the support of the grant S2009/esp-1594 from the 'Comunidad de Madrid' (Spain) and the grant MTM2009-14039-C06-03 from the Spanish government.

Table 1. Instances description: DFM stands for 'Density of the flow matrix'

Ins.	n	DFM (%)	Nb of x	Nb of z	Nb of y	Nb of constraints		
				GLL/KLB/XYL	R-III[a]	GLL/KLB	XYL	R-III
Chr18a	18	10.5	324	324	5202	360	684	648
Chr20a	20	9.5	400	400	7220	440	840	800
Chr22a	22	8.7	484	484	9702	528	1012	968
Chr25a	25	7.7	625	625	14400	675	1300	1250
P. ave.[b]	21	9.1	458	458	9131	501	959	917
Esc16a	16	29.7	256	256	9120	288	544	1248
Esc16b	16	71.9	256	256	22080	288	544	2976
Esc16c	16	39.8	256	256	12240	288	544	1664
Esc32a	32	14.5	1024	1024	73408	1088	2112	4800
Esc32b	32	21.1	1024	1024	107136	1088	2112	6976
Esc32c	32	25.6	1024	1024	129952	1088	2112	8448
P. ave.	24	33.8	640	640	58989	688	1328	4352
Kra30a	30	36.7	900	900	143550	960	1860	9960
Kra30b	30	36.7	900	900	143550	960	1860	9960
Kra32	32	32.2	1024	1024	163680	1088	2112	10624
P. ave.	31	35.2	941	941	150260	1003	1944	10181
Scr12	12	38.9	144	144	3696	168	312	696
Scr15	15	37.3	225	225	8820	255	480	1290
Scr20	20	31.0	400	400	23560	440	840	2520
P. ave.	16	35.7	256	256	12025	288	544	1502
Bur26a	26	77.7	676	676	167050	728	1404	13416
Bur26b	26	77.7	676	676	167050	728	1404	13416
Bur26c	26	74.3	676	676	156000	728	1404	12532
Bur26d	26	74.3	676	676	156000	728	1404	12532
Bur26e	26	68.8	676	676	144950	728	1404	11648
Bur26f	26	68.8	676	676	144950	728	1404	11648
Bur26g	26	78.8	676	676	169650	728	1404	13624
Bur26h	26	78.8	676	676	169650	728	1404	13624
P. ave.	26	74.9	676	676	159413	728	1404	12805
Nug21	21	62.1	441	441	57540	483	924	5796
Nug22	22	63.2	484	484	70686	528	1012	6776
Nug24	24	64.2	576	576	102120	624	1200	8928
Nug25	25	64.0	625	625	120000	675	1300	10050
Nug27	27	63.9	729	729	163566	783	1512	12636
Nug28	28	64.0	784	784	189756	840	1624	14112
Nug30	30	65.1	900	900	254910	960	1860	17640
P. ave.	25	63.8	648	648	136940	699	1347	10848
T. ave.[c]	24	48.3	620	620	100361	668	1288	7845

[a] In these tables, R-III signify IPQAPR-III.
[b] In this paper, P. ave. is the abbreviation for partial average.
[c] T. ave. is the abbreviation for total average.

Table 2. Quality of the mixed integer linear programming solution

Ins.	Opt. or B. known Cost	(Mixed) ILP							
		GLL		KBL		XYL		R-III	
		Cost	B&B/Opt. Gap(%)	Cost	B&B/Opt. Gap(%)	Cost	B&B/Opt. Gap(%)	Cost	B&B/Opt. Gap(%)
Chr18a	11098	11098	0/0	12938	83/17	11098	0/0	11098	0/0
Chr20a	2192	2192	0/0	2390	83/9	2192	0/0	2192	0/0
Chr22a	6156	6156	0/0	6732	88/9	6156	0/0	6156	0/0
Chr25a	3796	4162	21/10	5912	97/56	3920	15/3	3796	0/0
P. ave.	5811	5902	5/2	6993	88/23	5842	4/1	5811	0/0
Esc16a	68	68	0/0	68	30/0	68	0/0	68	36/0
Esc16b	292	292	20/0	292	68/0	292	15/0	292	100/0
Esc16c	160	160	41/0	160	81/0	160	34/0	162	78/1
Esc32a	130(B)[a]	140	72/8	154	96/18	142	72/9	244	100/88
Esc32b	168(B)	204	53/21	200	98/19	168	41/0	400	100/138
Esc32c	642(B)	642	44/0	644	96/ϵ[b]	642	44/0	716	100/12
P. ave.	243	251	38/5	253	78/6	245	34/2	314	86/40
Kra30a	88900	97460	29/10	96320	100/8	96090	28/8	115560	68/30
Kra30b	91420	94420	26/3	97230	100/6	93950	25/3	112660	67/23
Kra32	88700	98240	31/11	98300	100/11	93000	27/5	119490	70/35
P. ave.	89673	96707	29/8	97283	100/9	94347	27/5	115903	68/29
Scr12	31410	31410	0/0	31410	0/0	31410	0/0	31410	0/0
Scr15	51140	51140	0/0	54734	43/7	51140	0/0	51140	0/0
Scr20	110030	114272	20/4	119452	90/9	110030	16/0	118760	24/8
P. ave.	64193	65607	7/1	68532	44/5	64193	5/0	67103	8/3
Bur26a	5426670	5435370	2/ϵ	5435737	99/ϵ	5434439	2/ϵ	0	100/100
Bur26b	3817852	3827657	3/ϵ	3824650	99/ϵ	3828027	3/ϵ	4171639	12/9
Bur26c	5426795	5428356	2/ϵ	5428356	99/ϵ	5427052	2/ϵ	5802498	9/7
Bur26d	3821225	3821784	2/ϵ	3835732	100/ϵ	3821731	3/ϵ	0	100/100
Bur26e	5386879	5387166	1/ϵ	5400754	99/ϵ	5388552	1/ϵ	0	100/100
Bur26f	3782044	3782747	2/ϵ	3797282	99/ϵ	3782623	2/ϵ	0	100/100
Bur26g	10117172	10119059	1/ϵ	10117573	96/ϵ	10118719	1/ϵ	0	100/100
Bur26h	7098658	7099924	1/ϵ	7099097	98/ϵ	7098905	1/ϵ	7242138	5/2
P. ave.	5609662	5612758	2/ϵ	5617398	98/ϵ	5612506	2/ϵ	2152034	66/65
Nug21	2438	2522	25/3	2554	93/5	2476	23/2	2876	63/18
Nug22	3596	3716	31/3	3728	94/4	3632	29/1	4068	71/13
Nug24	3488	3640	25/4	3712	95/6	3526	23/1	4454	68/28
Nug25	3744	3890	25/4	3936	95/5	3760	23/ϵ	4662	67/25
Nug27	5234	5510	32/5	5602	97/7	5402	31/3	7180	75/37
Nug28	5166	5522	30/7	5510	96/7	5264	27/2	7254	74/40
Nug30	6124	6432	29/5	6470	98/6	6264	27/2	8038	72/31
P. ave.	4256	4462	28/5	4502	95/6	4332	26/2	5505	70/27
T. ave.	1464303	1465979	18/3	1467665	87/7	1465511	17/1	575127	57/34

[a] B means the value is the best known solution.
[b] ϵ means that the corresponding value is less than 0.5.

Table 3. Cplex CPU time and number of branch and bound nodes

Ins.	Cplex Time (Sec.)				Nb of Nodes ($\times 10^3$)			
	GLL	KBL	XYL	R-III	GLL	KBL	XYL	R-III
Chr18a	410	2806(M)[a]	391	7	486	2344	251	0.071
Chr20a	29	4134(M)	126	29	12	1722	36	0.413
Chr22a	162	4272(M)	207	21	108	999	86	0.230
Chr25a	1586(M)	2790(M)	5869(M)	274	858	656	1639	5.164
P. ave.	547	3500	1648	83	366	1139	503	0.712
Esc16a	1366	2955(M)	827	14400(*)[b]	2905	6154	1158	155.689
Esc16b	8033(M)	6879(M)	13197(M)	14400(*)	4098	3549	6940	1.590
Esc16c	3577(M)	3092(M)	5086(M)	14400(*)	3993	3893	4204	55.170
Esc32a	14400(*)	9976(M)	14400(*)	14400(*)	975	1153	661	0.140
Esc32b	11437(M)	9395(M)	14400(*)	14400(*)	1169	1171	707	0.017
Esc32c	14400(*)	7025(M)	14400(*)	14400(*)	988	1168	598	0.031
P. ave.	8869	6554	10385	14400(*)	2354	2848	2378	38.768
Kra30a	5918(M)	9632(M)	9429(M)	14400(*)	449	511	559	0.043
Kra30b	6482(M)	10053(M)	14400(*)	14400(*)	454	520	538	0.029
Kra32	7920(M)	10913(M)	14400(*)	14400(*)	420	503	353	0.025
P. ave.	6773	10199	12743	14400(*)	441	511	483	0.017
Scr12	8	207	10	107	16	475	14	3.791
Scr15	252	2408(M)	310	1387	298	2276	251	10.408
Scr20	2090	3203	6369	14400(*)	1148	1212	1403	18.259
P. ave.	783	1939	2230	5298	487	1321	556	9.858
Bur26a	14400(*)	14400(*)	14400(*)	14400(*)	749	900	618	0.001
Bur26b	14400(*)	14400(*)	14400(*)	14400(*)	673	1071	619	0.001
Bur26c	14400(*)	14400(*)	14400(*)	14400(*)	807	1142	781	0.001
Bur26d	14400(*)	14400(*)	14400(*)	14400(*)	662	1067	636	0.001
Bur26e	14400(*)	14400(*)	14400(*)	14400(*)	859	1448	790	0.001
Bur26f	14400(*)	14400(*)	14400(*)	14400(*)	711	1505	828	-
Bur26g	14400(*)	14400(*)	14400(*)	14400(*)	756	864	939	0.001
Bur26h	14400(*)	14400(*)	14400(*)	14400(*)	847	822	626	0.002
P. ave.	14400(*)	14400(*)	14400(*)	14400(*)	758	1102	729	0.001
Nug21	6199(M)	4622(M)	5816(M)	14400(*)	1732	1018	1451	0.511
Nug22	4374(M)	4937(M)	14400(*)	14400(*)	874	931	1345	0.236
Nug24	4155(M)	5648(M)	14400(*)	14400(*)	665	714	1070	0.055
Nug25	4739(M)	6527(M)	14400(*)	14400(*)	610	653	718	0.015
Nug27	6516(M)	8190(M)	14400(*)	14400(*)	511	566	614	0.004
Nug28	6931(M)	9262(M)	14400(*)	14400(*)	478	515	511	0.003
Nug30	11304(M)	11529(M)	14400(*)	14400(*)	413	452	370	0.001
P. ave.	6317	7245	13174	14400(*)	755	692	868	0.118
T. ave.	7661	8247	10362	11672	959	1354	1010	8.397

[a]Because of out of memory, Cplex stopped before the CPU time limit was reached.
[b]The CPU time limit, 14400 seconds, was reached.

Table 4. Quality of the LP bound

Ins.	Linear relaxation								LP CPU Time (Sec.)			
	GLL		KBL		XYL		R-III					
	LPcost	LPgap (%)	LPcost	LPgap (%)	LPcost	LPgap (%)	LPcost	LPgap (%)	GLL	KBL	XYL	R-III
Chr18a	6885	38	0	100	6885	38	9515	14	ϵ^a	1	1	ϵ
Chr20a	2150	2	0	100	2150	2	2156	2	ϵ	ϵ	1	ϵ
Chr22a	5927	4	0	100	5927	4	5994	3	ϵ	ϵ	1	1
Chr25a	2787	27	0	100	2787	27	3272	14	ϵ	ϵ	1	1
P. ave.	4437	18	0	100	4437	18	5234	8	ϵ	ϵ	1	1
Esc16a	38	44	0	100	38	44	0	100	1	1	ϵ	1
Esc16b	220	25	0	100	220	25	0	100	1	1	ϵ	6
Esc16c	83	48	0	100	83	48	0	100	1	1	ϵ	2
Esc32a	35	73	0	100	35	73	0	100	1	1	ϵ	94
Esc32b	96	43	0	100	96	43	0	100	1	1	1	286
Esc32c	350	45	0	100	350	45	0	100	1	1	1	403
P. ave.	137	46	0	100	137	46	0	100	1	1	ϵ	132
Kra30a	68360	23	0	100	68360	23	36864	59	1	1	2	535
Kra30b	69065	24	0	100	69065	24	36700	60	1	1	3	323
Kra32	67390	24	0	100	67390	24	36400	59	1	ϵ	3	381
P. ave.	68272	24	0	100	68272	24	36655	59	1	1	3	413
Scr12	27858	11	0	100	27858	11	25474	19	ϵ	ϵ	ϵ	ϵ
Scr15	44737	13	0	100	44737	13	40026	22	ϵ	ϵ	ϵ	2
Scr20	86766	21	0	100	86766	21	75420	31	1	ϵ	ϵ	7
P. ave.	53120	15	0	100	53120	15	46973	24	ϵ	ϵ	ϵ	3
Bur26a	5315337	2	0	100	5315268	2	5288658	3	1	ϵ	2	2495
Bur26b	3714887	3	0	100	3714819	3	3692082	3	1	ϵ	4	3592
Bur26c	5312148	2	0	100	5312146	2	5262370	3	1	ϵ	4	3916
Bur26d	3711824	3	0	100	3711820	3	3673730	4	1	ϵ	3	2625
Bur26e	5307278	1	0	100	5307214	1	5272546	2	1	ϵ	4	2265
Bur26f	3707055	2	0	100	3707002	2	3680834	3	1	ϵ	3	1886
Bur26g	9978615	1	0	100	9978473	1	9905912	2	1	1	3	1445
Bur26h	6973656	2	0	100	6973477	2	6915448	3	1	1	3	2682
P. ave.	5502600	2	0	100	5502527	2	5461448	3	1	ϵ	3	2613
Nug21	1833	25	0	100	1833	25	1054	57	ϵ	1	1	113
Nug22	2483	31	0	100	2483	31	1188	67	ϵ	1	1	95
Nug24	2676	23	0	100	2676	23	1430	59	ϵ	1	1	251
Nug25	2870	23	0	100	2870	23	1532	59	ϵ	1	2	536
Nug27	3701	29	0	100	3701	29	1819	65	ϵ	1	1	1688
Nug28	3786	27	0	100	3786	27	1890	63	ϵ	1	2	992
Nug30	4539	26	0	100	4539	26	2218	64	1	1	4	1934
P. ave.	3127	26	0	100	3127	26	1590	62	ϵ	1	2	801
T. ave.	1433079	21	0	100	1433060	21	1418533	43	1	ϵ	2	921

[a] ϵ means that the corresponding value is less than 0.5.

References

[1] L. Kaufman and F. Broeckx. An algorithm for the quadratic assignment problem using benders' decomposition. *European Journal of Operational Research*, 2(3):204–211, 1978.

[2] T. C. Koopmans and M. J. Beckmann. Assignment problems and the location of economic activities. *Econometrica*, 25(1):53–76, 1957.

[3] E. L. Lawler. The quadratic assignment problem. *Management Science*, 9(4):586–599, 1963.

[4] S. Sahni and T. Gonzalez. P-complete approximation problems. *Journal of the Association of Computing Machinery*, 23(3):555–565, 1976.

[5] E. M. Loiola, N. M. M. Abreu, P. O. Boaventura-Netto, P. Hahn, and T. Querido. A survey for the quadratic assignment problem. *European Journal of Operational Research*, 176(2):657–690, 2007.

[6] P. M. Pardalos, F. Rendl, and H. Wolkowicz. The quadratic assignment problem: A survey and recent developments. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science, American Mathematical Society*, 16:1–42, 1994.

[7] L. Steinberg. The backboard wiring problem: A placement algorithm. *SIAM Reviews*, 3(1):37–50, 1961.

[8] M. A. Pollatschek, N. Gershoni, and Y. T. Radday. Optimization of the typewriter keyboard by simulation. *Angewandte Informatik*, 17(0):438–439, 1976.

[9] A. M. Geoffrion and G. W. Graves. Scheduling parallel production lines with changeover costs: Practical applications of a quadratic assignment/lp approach. *Operations Research*, 24(4):595–610, 1976.

[10] M. J. Brusco and S. Stahl. Using quadratic assignment methods to generate initial permutations for least-squares unidimensional scaling of symmetric proximity matrices. *Journal of Classification*, 17(2):197–223, 2000.

[11] G. G. Polak. On a special case of the quadratic assignment problem with an application to storage and retrivel devices. *Annals of Operations Research*, 138(1), 2005.

[12] P. Mills, E. Tsang, and J. Ford. Applying and extended guided local search to the quadratic assignment problem. *Annals of Operations Research*, 118(1-4):121–135, 2003.

[13] E. Çela. *The Quadratic Assignment Problem: Theory and Algorithms*. Kluwer Academic Publishers, London, 1998.

[14] K. M. Anstreicher. Recent advances in the solution of quadratic assignment problems. *Mathematical Programming*, 97(1-2):27–42, 2003.

[15] C. W. Commander. A survey of the quadratic assignment problem, with applications. *Morehead Electronic Journal of Applicable Mathematics*, 4(MATH-2005-01):1–15, 2005.

[16] Z. Drezner, P. M. Hahn, and É. D. Taillard. Recent advances for the quadratic assignment problem with special emphasis on instances that are difficult for meta-heuristic methods. *Annals of Operations Research*, 139(1), 2005.

[17] R. Burkard, M. Dell'Amico, and S. Martello. *Assignment Problems*, pages 203–304. Society for Industrial and Applied Mathematics, Philadelphia, 2009.

[18] M. Bayat and M. Sedghi. *Facility Location: Concepts, Models, Algorithms and Case Studies*, chapter Quadratic Assignment Problem, pages 111–143. Physica-Verlag, Heidelberg, 2009.

[19] W. P. Adams and T. A. Johnson. Improved linear programming-based lower bounds for the quadratic assignment problem. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science, American Mathematical Society*, 16:43–75, 1994.

[20] A. M. Frieze and J. Yadegar. On the quadratic assignment problem. *Discrete Applied Mathematics*, 5(1):89–98, 1983.

[21] P. Hahn and T. Grant. Lower bounds for the quadratic assignment problem based upon a dual formulation. *Operations Research*, 46(6):912–922, 1998.

[22] Y. Xia and Y. X. Yuan. A new linearization method for quadratic assignment problem. *Optimization Methods and Software*, 21(5):803–816, 2006.

[23] Y. Xia. Improved gilmore-lawler bound for quadratic assignment problem. *Chinese Journal of Engineering Mathematics*, 24(3):401–413, 2007.

[24] Y. Xia. Gilmore-lawler bound of quadratic assignment problem. *Frontiers of Mathematics in China*, 3(1):109–118, 2008.

[25] P. C. Gilmore. Optimal and suboptimal algorithms for the quadratic assignment problem. *Journal of the Society for Industrial and Applied Mathematices*, 10(2):305–313, 1962.

[26] R. E. Burkard. Locations with spatial interactions: The quadratic assignment problem. In P.B.Mirchandani and R.L.Francis, editors, *Discrete Location Theory*, pages 387–437. Wiley, 1991.

[27] R. E. Burkard, S. E. Karisch, and F. Rendl. Qaplib-a quadratic assignment problem library. *Journal of Global Optimization*, 10(4):391– 403, 1997. http://www.seas.upenn.edu/qaplib/.

[28] H. Z. Zhang, C. Beltran-Royo, and M. Constantino. Effective formulation reductions for the quadratic assignment problem. *Computers and Operations Research*, (DOI information: 10.1016/j.cor.2010.02.001), 2010.

[29] M. Baotic. Matlab interface for cplex, http://control.ee.ethz.ch/ hybrid/cplexint.php 2004.